

# High probability state transfer in spin-1/2 chains: Analytical and numerical approaches

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## Abstract

This article is devoted to the development of analytical and numerical approaches to the problem of the end-to-end quantum state transfer along the spin-1/2 chain using two methods: (a) a homogeneous spin chain with weak end bonds and equal Larmor frequencies and (b) a homogeneous spin chain with end Larmor frequencies different from inner ones. A tridiagonal matrix representation of the XY Hamiltonian with nearest neighbor interactions relevant to the quantum state transfer is exactly diagonalized for a combination of the above two methods. In order to take into account interactions of the remote spins we used numerical simulations of the quantum state transfer in ten-node chains. We compare the state transfer times obtained using the two above methods for chains governed by the both XY and XXZ Hamiltonians and using both nearest neighbor and all node interactions.

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## I. INTRODUCTION

The problem of quantum state transfer along the spin chain [1] acquires significance in the study of quantum communication systems. Several models of the end-to-end quantum state transfer along the spin-1/2 chain were suggested [2–9]. All these models are based on using either inhomogeneous chains with different coupling constants between nearest neighbors (NN) or homogeneous chains with different Larmor frequencies on the chain nodes. Seminal works [2–4, 6] use NN interactions in the inhomogeneous symmetrical chain. All coupling constants of such chains have fixed values, allowing one to transfer the state during the time interval independent of the length of the chain. The method proposed in refs.[5, 6] and developed in refs.[7, 8] is based on the weak end bonds in the chain when interactions of all remote spins are taken into account (the weak end bonds method (WEBM)). Transfer time increases with the length of the chain in this case.

Since it is very difficult to construct inhomogeneous chains, an alternative method of state transfer along the spin chain has been proposed recently [9]. The effect of the high probability state transfer (HPST) along the homogeneous spin-1/2 chain is achieved here because of the specially tailored external inhomogeneous magnetic field. In the simplest case one needs only two nonzero end Larmor frequencies and zero inner ones (end Larmor frequencies method (ELFM)).

The problem of quantum state transfer requires solving some physical questions. In particular, it is not clear whether the presence of inner nodes (so-called "body" of the chain) improves parameters of state transfer (transfer time and probability). How will the transfer time be changed if we fix end nodes and remove the body? Even this very simple question requires a special study and will be considered for two methods of state transfer (WEBM and ELFM) applied to the spin-1/2 chain governed by either an XY or XXZ Hamiltonian with dipole-dipole interactions.

In this article, we develop an approach to the exact diagonalization of the XY-Hamiltonian with NN interactions for the combination of WEBM and ELFM, Fig.1. We show also the formal equivalence of the spin dynamics governed by the XY and XXZ Hamiltonians in the specially tailored inhomogeneous external magnetic field in the problem of quantum state transfer with a single excited spin.

We perform the numerical simulations of quantum state transfer along the ten-node

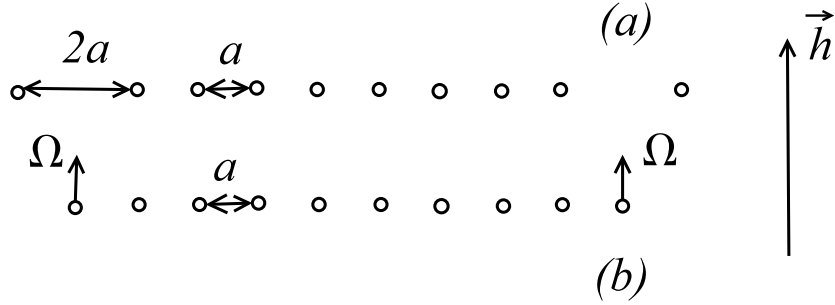


FIG. 1: Two methods of the state transfer along the spin-1/2 chain;  $a$  is the distance between nearest neighbors in the body of the chain: (a) WEBM, all Larmor frequencies are zeros; (b) ELFM along the homogeneous chain; the end node Larmor frequencies equal to  $\Omega$ , while inner ones are zeros.

chain with dipole-dipole interactions governed by different Hamiltonians (XY or XXZ) using different methods of state transfer (WEBM or ELFM) and different types of interactions (NN or all node interactions). We concentrate on the following problems in numerical simulations:

1. Study the influence of the inner nodes on the transfer time over the given distance  $L$ . We will show that inner nodes may either decrease or increase the transfer time in dependence on the Hamiltonian, the method of the state transfer and the type of interactions.
2. Compare the end-to-end transfer times along the chains governed by different Hamiltonians with different methods of the state transfer and different types of interactions.

Thus, the structure of this paper is following. We give some details regarding the spin dynamics in Sec.II. The analytical description of the spin dynamics along the chain governed by the XY Hamiltonian with NN interactions using the combination of WEBM and ELFM is given in Sec. III. The numerical simulations of the quantum state transfer corresponding to different Hamiltonians, methods of the state transfer and types of interactions are represented in Sec.IV. We briefly summarize our results in the concluding section, Sec.V.

## II. SINGLE EXCITED SPIN DYNAMICS

Consider the spin dynamics along the spin-1/2 chains governed by either XY or XXZ Hamiltonian in the inhomogeneous external magnetic field. These Hamiltonians read:

$$\mathcal{H}_{XY} = \sum_{\substack{i,j=1 \\ j>i}}^N D_{i,j} (I_{i,x}I_{j,x} + I_{i,y}I_{j,y}) + \sum_{i=1}^N \Omega_i^{XY} I_{i,z}, \quad D_{i,j} = \frac{\gamma^2 \hbar}{2r_{i,j}^3}, \quad (1)$$

$$\mathcal{H}_{XXZ} = \sum_{\substack{i,j=1 \\ j>i}}^N D_{i,j} (I_{i,x}I_{j,x} + I_{i,y}I_{j,y} - 2I_{i,z}I_{j,z}) + \sum_{i=1}^N \Omega_i^{XXZ} I_{i,z}. \quad (2)$$

Here  $D_{i,j}$  is the coupling constant between the  $i$ th and the  $j$ th nodes. Hereafter we will study the one-dimensional spin-1/2 chains and use the dimensionless time  $\tau$ , distances  $\xi_{n,m}$ , coupling constants  $d_{n,m}$  and Larmor frequencies  $\omega_n$ , defined as follows:

$$\begin{aligned} \tau &= D_{1,2}t, \quad \xi_{n,m} = \frac{r_{n,m}}{r_{1,2}}, \quad d_{n,m} = \frac{D_{n,m}}{D_{1,2}} = \frac{1}{\xi_{n,m}^3}, \quad d_{1,2} = 1, \\ \omega_n^{XY} &= \frac{\Omega_n^{XY}}{D_{1,2}}, \quad \omega_n^{XXZ} = \frac{\Omega_n^{XXZ}}{D_{1,2}}. \end{aligned} \quad (3)$$

Using definitions (3), the Hamiltonians (1) and (2) may be written as follows:

$$\mathcal{H}_{XY} = D_1 \tilde{\mathcal{H}}_{XY}, \quad \tilde{\mathcal{H}}_{XY} = \sum_{\substack{i,j=1 \\ j>i}}^N d_{j,i} (I_{i,x}I_{j,x} + I_{i,y}I_{j,y}) + \sum_{i=1}^N \omega_i^{XY} I_{i,z}, \quad (4)$$

$$\mathcal{H}_{XXZ} = D_1 \tilde{\mathcal{H}}_{XXZ}, \quad \tilde{\mathcal{H}}_{XXZ} = \sum_{\substack{i,j=1 \\ j>i}}^N d_{j,i} (I_{i,x}I_{j,x} + I_{i,y}I_{j,y} - 2I_{i,z}I_{j,z}) + \sum_{i=1}^N \omega_i^{XXZ} I_{i,z}. \quad (5)$$

In order to characterize the effectiveness of the quantum state transfer from the first to the  $N$ th node, the fidelity  $F(\tau)$  has been introduced [1]:

$$F(\tau) = \frac{|f_{1N}(\tau)| \cos \Gamma}{3} + \frac{|f_{1N}(\tau)|^2}{6} + \frac{1}{2}, \quad \Gamma = \arg f_{1,N}, \quad (6)$$

where  $f_{n,m}$  is a transfer amplitude:

$$f_{n,m}(\tau) = \langle m | e^{-i\tilde{\mathcal{H}}\tau} | n \rangle = \sum_{j=1}^N u_{n,j} u_{m,j} e^{-i\lambda_j \tau/2}, \quad f_{n,m} = f_{m,n}. \quad (7)$$

Here  $u_{i,j}$ ,  $i, j, = 1, \dots, N$ , are components of the normalized eigenvector  $u_j$  corresponding to the eigenvalue  $\lambda_j$  of the matrix  $D$ ,  $H = \frac{1}{2}D$ :  $Du_j = \lambda_j u_j$ , where  $H$  is the matrix representation of  $\tilde{\mathcal{H}}$ .

### A. Equivalence of XY and XXZ Hamiltonians

It is simple to show the equivalence of the single excited spin dynamics governed by the XY and XXZ Hamiltonians. In fact, two Hamiltonians (4) and (5) differ by the diagonal parts, i.e.

$$\Delta\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{XY} - \tilde{\mathcal{H}}_{XXZ} = \sum_{i=1}^N (\omega_i^{XY} - \omega_i^{XXZ}) I_{i,z} + \sum_{\substack{i,j=1 \\ j>i}}^N 2d_{i,j} I_{i,z} I_{j,z}. \quad (8)$$

Since we consider the single excited state transfer and both Hamiltonians commutes with  $I_z$  ( $z$  projection of the total spin), the spin dynamics is described by only one block of the Hamiltonian which may be written on the basis of  $N$  eigenvectors  $|n\rangle$ ,  $n = 1, \dots, N$ , where notation  $|n\rangle$  means that the  $n$ th node is excited, i.e. directed opposite to the external magnetic field. Then the matrix representation  $\Delta\tilde{H}$  of  $\Delta\tilde{\mathcal{H}}$  reads [10, 11] as follows:

$$\begin{aligned} \Delta\tilde{H} &= \omega^{XY} - \omega^{XXZ} - A + \Gamma E_N, \\ \omega^{XY} &= \text{diag}(\omega_1^{XY}, \dots, \omega_N^{XY}), \quad \omega^{XXZ} = \text{diag}(\omega_1^{XXZ}, \dots, \omega_N^{XXZ}), \quad A = \text{diag}(A_1, \dots, A_N), \\ A_n &= \sum_{\substack{i=1 \\ i \neq n}}^N d_{i,n}, \quad \Gamma = \frac{1}{2} \sum_{\substack{i,j=1 \\ j>i}}^N d_{i,j} + \frac{1}{2} \sum_{i=1}^N (\omega_i^{XXZ} - \omega_i^{XY}). \end{aligned} \quad (9)$$

Thus both Hamiltonians are equal to each other if  $\Delta\tilde{H} = 0$ . Since the number of the diagonal elements in  $\Delta\tilde{H}$  coincides with the number of Larmor frequencies in each Hamiltonian, this condition may be satisfied by the proper choice of the Larmor frequencies in either the XY or XXZ Hamiltonian. This conclusion remains correct for an arbitrary spin-1/2 system. Nevertheless, we consider XY and XXZ Hamiltonians independently as far as there is no a simple method of generation of the inhomogeneous magnetic field.

Remember that  $\Gamma$  may be put to zero by the proper choice of the constant homogeneous magnetic field [1]. For this reason, hereafter we consider the probability of the state transfer,  $P(\tau) = |f_{1N}(\tau)|^2$ , instead of the fidelity as a characteristic of the state transfer effectiveness.

### III. THE EXPLICIT SOLUTION FOR THE SPIN DYNAMICS IN THE CHAIN GOVERNED BY THE XY-HAMILTONIAN WITH NN INTERACTIONS

In this section, we develop the ideas of Ref.[10] and give an analytical description of the single excited spin dynamics in the chain governed by the XY Hamiltonian with NN

interactions combining both WEBM and ELFM. In other words, one has to substitute

$$d_{i,i+1} = \begin{cases} \delta, & 1 < i < N-1 \\ 1, & i = 1, N-1 \end{cases}, \quad \omega_i^{XY} = \begin{cases} 0, & 1 < i < N \\ \omega, & i = 1, N \end{cases}. \quad (10)$$

into the Hamiltonian (4), which reads:

$$\begin{aligned} \tilde{\mathcal{H}}_{XY} = & \quad (11) \\ \sum_{i=2}^{N-2} \frac{\delta}{2} (I_i^+ I_{i+1}^- + I_i^- I_{i+1}^+) + \frac{1}{2} (I_1^+ I_2^- + I_1^- I_2^+ + I_{N-1}^+ I_N^- + I_{N-1}^- I_N^+) + \omega(I_{1,z} + I_{N,z}). \end{aligned}$$

Since we are interested in the single excited spin dynamics, the Hamiltonian has  $N \times N$  matrix representation (see Sec.II), which reads (up to the scalar term, which is not important in our case and may be removed by adding a proper homogeneous constant external magnetic field):

$$H = \frac{1}{2}D, \quad D = \begin{pmatrix} 2\omega & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \delta & \cdots & 0 & 0 \\ 0 & \delta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2\omega \end{pmatrix}. \quad (12)$$

First, one has to find eigenvalues and eigenvectors of  $D$ . For this purpose, we solve the equation

$$Du_j = \lambda_j u_j, \quad u_j = (u_{1j}, \dots, u_{Nj})^T \quad (13)$$

for the components of the eigenvectors  $u_j$ , where  $\lambda_j$  are the solutions to the characteristic equation

$$\det(D - \lambda E_N) = 0, \quad (14)$$

and  $E_N$  is an  $N \times N$  unit matrix. The system (13) may be written as follows:

$$2\omega u_{1j} + u_{2j} = \lambda_j u_{1j}, \quad (15)$$

$$u_{1j} + \delta u_{3j} = \lambda_j u_{2j}, \quad (16)$$

$$\delta u_{(k-1)j} + \delta u_{(k+1)j} = \lambda_j u_{kj}, \quad 3 \leq k \leq N-2 \quad (17)$$

$$\delta u_{(N-2)j} + u_{Nj} = \lambda_j u_{(N-1)j}, \quad (18)$$

$$u_{(N-1)j} + 2\omega u_{Nj} = \lambda_j u_{Nj}. \quad (19)$$

Eqs.(17) are known to have the following solution

$$u_{kj} = C_{1j}e^{-ikp_j} + C_{2j}e^{ikp_j}, \quad 2 \leq k \leq N-1, \quad (20)$$

$$\lambda_j = 2\delta \cos p_j. \quad (21)$$

Then eq.(14) becomes equivalent to the compatibility condition of the system of four equations, eqs. (15,16,18,19) for any  $j$ . After some transformations, this compatibility condition reads

$$4i \left( \delta(2\omega - 2\delta \cos p) \cos \frac{N-1}{2}p + \cos \frac{N-3}{2}p \right) \times \quad (22)$$

$$\left( \delta(2\omega - 2\delta \cos p) \sin \frac{N-1}{2}p + \sin \frac{N-3}{2}p \right) = 0.$$

One can show that this equation can be treated as the  $N$  degree polynomial equation for the variable  $X = \cos p$ , and consequently, it has  $N$  roots. All roots can be separated into two sets of  $N_1$  and  $N - N_1$  roots, which are solutions to one of two following equations:

$$\delta(2\omega - 2\delta \cos p_j) \cos \frac{N-1}{2}p_j + \cos \frac{N-3}{2}p_j = 0, \quad j = 1, \dots, N_1, \quad N_1 = N - \left[ \frac{N}{2} \right] \quad (23)$$

$$\delta(2\omega - 2\delta \cos p_j) \sin \frac{N-1}{2}p_j + \sin \frac{N-3}{2}p_j = 0, \quad j = N_1 + 1, \dots, N.$$

Here  $[a]$  means the integer part of  $a$ . Thus, using eqs.(23), expressions for the normalized eigenvector components can be written as follows:

$$u_{1j} = \begin{cases} \delta A_j \cos \frac{(N-1)p_j}{2}, & j = 1, \dots, N_1 \\ \delta A_j \sin \frac{(N-1)p_j}{2}, & j = N_1 + 1, \dots, N \end{cases}, \quad (24)$$

$$u_{kj} = \begin{cases} A_j \cos \frac{(N+1-2k)p_j}{2}, & k = 2, \dots, N-1, \quad j = 1, \dots, N_1 \\ A_j \sin \frac{(N+1-2k)p_j}{2}, & k = 2, \dots, N-1, \quad j = N_1 + 1, \dots, N \end{cases},$$

$$u_{Nj} = \begin{cases} \delta A_j \cos \frac{(N-1)p_j}{2}, & j = 1, \dots, N_1 \\ -\delta A_j \sin \frac{(N-1)p_j}{2}, & j = N_1 + 1, \dots, N \end{cases},$$

where

$$A_j = \begin{cases} \left( \frac{N-2}{2} + \delta^2(1 + \cos(N-1)p_j) + \frac{\sin(N-2)p_j}{2 \sin p_j} \right)^{-1/2}, & j = 1, \dots, N_1 \\ \left( \frac{N-2}{2} + \delta^2(1 - \cos(N-1)p_j) - \frac{\sin(N-2)p_j}{2 \sin p_j} \right)^{-1/2}, & j = N_1 + 1, \dots, N \end{cases}. \quad (25)$$

Then the probability of the end-to-end excited state transfer reads:

$$P(\tau) = \left| \langle N | e^{-i\tilde{H}_{XY}\tau} | 1 \rangle \right|^2 = \left| \sum_{j=1}^N u_{Nj} u_{1j} e^{-i\tau\lambda_j/2} \right|^2 = \delta^4 \left| \sum_{j=1}^{N_1} A_j^2 \cos^2(N-1) \frac{p_j}{2} e^{-i\delta\tau \cos p_j} - \sum_{j=N_1+1}^N A_j^2 \sin^2(N-1) \frac{p_j}{2} e^{-i\delta\tau \cos p_j} \right|^2, \quad (26)$$

which will be used in those examples of Sec.IV where NN interactions are considered.

### A. Spin chain of four nodes

The derived formulas become most simple in the case  $N = 4$ . The eigenvalues (21) have the following explicit forms:

$$\begin{aligned} \lambda_1 &= \frac{2\omega - \delta + \sqrt{(2\omega + \delta)^2 + 4}}{2}, & \lambda_2 &= \frac{2\omega - \delta - \sqrt{(2\omega + \delta)^2 + 4}}{2}, \\ \lambda_3 &= \frac{2\omega + \delta + \sqrt{(2\omega - \delta)^2 + 4}}{2}, & \lambda_4 &= \frac{2\omega + \delta - \sqrt{(2\omega - \delta)^2 + 4}}{2}. \end{aligned} \quad (27)$$

Formula (26) reduces to the following one:

$$P(\tau) = \left| \frac{1}{2} e^{-i\delta\tau/4} \left\{ \cos(\tau\beta(\omega, \delta)/4) - i \frac{2\omega - \delta}{\beta(\omega, \delta)} \sin(\tau\beta(\omega, \delta)/4) \right\} - \frac{1}{2} e^{i\delta\tau/4} \left\{ \cos(\tau\alpha(\omega, \delta)/4) - i \frac{2\omega + \delta}{\alpha(\omega, \delta)} \sin(\tau\alpha(\omega, \delta)/4) \right\} \right|^2, \quad (28)$$

where

$$\alpha(\omega, \delta) = \sqrt{(2\omega + \delta)^2 + 4}, \quad \beta(\omega, \delta) = \sqrt{(2\omega - \delta)^2 + 4}. \quad (29)$$

Let us demonstrate that the perfect state transfer is possible at some time moment  $\tau_0$  if the parameters  $\omega$  and  $\delta$  have been properly chosen. First, we remark that perfect state transfer is achieved if

$$|\cos(\tau_0\delta/4)| = 1, \cos(\tau_0\alpha(\omega, \delta)/4) = \pm 1, \cos(\tau_0\beta(\omega, \delta)/4) = \mp 1, \quad (30)$$

or

$$|\sin(\tau_0\delta/4)| = 1, \cos(\tau_0\alpha(\omega, \delta)/4) = \pm 1, \cos(\tau_0\beta(\omega, \delta)/4) = \pm 1. \quad (31)$$

Eqs.(30) and (31) are equivalent to the following pair of complete systems of algebraic equations (remember that  $\tau_0$ ,  $\delta$ ,  $\alpha$ , and  $\beta$  must be positive, while  $\omega$  must be real):

$$\tau_0\delta = 4\pi n_1, \quad \tau_0\alpha(\omega, \delta) = 4\pi n_2, \quad \tau_0\beta(\omega, \delta) = 4\pi n_3, \quad (32)$$



$$n_1 = 1, 2, \dots, n_2 = 1, 2, \dots, n_2 - n_3 = \pm 1, \pm 3, \pm 5, \dots,$$

or

$$\tau_0 \delta = \pi(4n_1 + 2), \tau_0 \alpha(\omega, \delta) = 4\pi n_2, \tau_0 \beta(\omega, \delta) = 4\pi n_3, \quad (33)$$

$$n_1 = 0, 1, \dots, n_2 = 1, 2, \dots, n_2 - n_3 = 0, \pm 2, \pm 4, \dots$$

If  $\omega = 0$  (WEBM), then we obtain alternating spin chain of four nodes. In this case,  $\alpha(0, \delta) = \beta(0, \delta) = \sqrt{\delta^2 + 4}$  so that system (32) becomes inconsistent, while system (33) acquires the following form:

$$\tau_0 \delta = \pi(4n_1 + 2), \tau_0 \alpha(0, \delta) \equiv \tau_0 \sqrt{\delta^2 + 4} = 4\pi n_2, \quad (34)$$

$$n_1 = 0, 1, \dots, n_2 - n_1 = 1, 2, \dots,$$

which agrees with the results obtained for the alternating spin-1/2 chain of four nodes [10]. The relation between  $n_1$  and  $n_2$  in eq.(34) is a consequence of the inequality  $\delta < \alpha(0, \delta)$ .

#### IV. NUMERICAL SIMULATIONS OF THE SPIN DYNAMICS.

Now we represent results of the numerical simulations of the end-to-end single excited state transfer along the spin-1/2 chain of ten nodes. Considering the problem of state transfer between two end nodes separated by the dimensionless distance  $L = \xi_{1,N}$ , first, we recall the simplest model of two spins with the distance  $L$  between them (Sec.IV A). Next, in Sec.IV B, we find end-to-end transfer times over the same distance  $L$  along the ten-node chain using both XY and XXZ Hamiltonians, with both WEBM and ELFM using both NN and all node interactions. In particular, we study the problem whether the presence of inner nodes reduces the time interval required for state transfer over the distance  $L$  in comparison with the above simplest system of two interacting spins. It will be shown that the answer is not always positive.

Note that we solve the optimization problem for the case of all node interactions (i.e., we find such parameters of the spin chain which provide the end-to-end HPST during as short as possible a time interval). Then, we consider the spin dynamics using NN interactions and keeping the same parameters of the spin chain. This allows us to see whether NN interactions give different transfer times in comparison with the case of all node interactions.

### A. The state transfer over the distance $L$

It is well known [2] that the quantum state can be perfectly transferred between two spin-1/2 nodes separated by the distance  $L = \xi_{1,N}$  during the dimensionless time interval  $\tau_2$  such that (see eq.(3))

$$\tau_2(L) = \frac{\pi}{d_{1,N}} = L^3\pi. \quad (35)$$

However the time interval  $\tau_2$  increases with an increase in  $L$  so that the direct node-to-node quantum state transfer over the long distance  $L$  becomes impossible because of the quantum decoherence. It is assumed that the set of inner nodes placed between the above two nodes may help one to overcome the problem of decoherence. However, it seemed that these nodes were not always helpful.

We will use the dimensionless time interval  $\tau_2(L)$  as a characteristics of the  $N$ -node chain. We say that the inner nodes in the  $N$ -node chain are usefull if the end-to-end transfer time along this chain is less than  $\tau_2$ .

It is remarkable that the parameter  $\tau_2(L)$  depends only on  $L$  and does not depend on the Hamiltonians and methods of state transfer. In fact, the XY Hamiltonian coincides with the XXZ Hamiltonian up to the scalar term in the matrix representation (see eq.(9) with  $\Omega_1^{XY} = \Omega_N^{XY}$ ,  $\Omega_1^{XXZ} = \Omega_N^{XXZ}$ ,  $N = 2$ ) as far as the problem of the single quantum state transfer along the two-node chain is considered. This term does not effect on the probability of state transfer.

### B. The ten-node chain

We represent results of the numerical simulations of the end-to-end state transfer along the ten-node chain using both WEBM and ELFM. Considering WEBM, we will use parameter  $\delta$  introduced in eq.(10) and zero Larmor frequencies. Choosing the value of  $\delta$ , we follow the ideology of ref.[7], where the distance between the end nodes and the body of the chain is twice as long as the distance between neighbors in the body. Thus we take  $\delta = 8$ , see Fig.1a with  $a = 1$ . However, we have observed that all conclusions remain valid for different values of  $\delta$  as well. Considering ELFM, we take  $\delta = 1$  and nonzero end Larmor frequencies  $\omega$ , while all inner Larmor frequencies are zeros, see Fig.1b with  $a = 1$  and  $\Omega = \omega$ .

Results of our numerical simulations are represented in Figs.2-5, namely, the functions  $P(\tau)$  obtained using WEBM for the spin-1/2 chains governed by the XY and XXZ Hamiltonians are shown in Figs.2 and 3 respectively; similarly, functions  $P(\tau)$  obtained using the ELFM for the spin chains governed by the XY and XXZ Hamiltonians are shown in Figs.4 and 5, respectively. Dynamics with both all node interactions and NN interactions are represented therein. Parameters  $T^{(an)}$ ,  $P^{(an)} = P(T^{(an)})$  and  $T^{(nn)}$ ,  $P^{(nn)} = P(T^{(nn)})$  are the end-to-end transfer times and probabilities of these transfers in the case of all node interactions and NN interactions respectively. In the following, we collect the results of analysis of all figures 2-5.

(i) Let  $T^{(WEBM)}$  and  $T^{(ELFM)}$  be the transfer times corresponding to WEBM and ELFM respectively. Comparing the transfer times shown in Figs.2-5 with appropriate interval  $\tau_2$  we conclude that the inner nodes do not always reduce the transfer time in comparison with  $\tau_2$  (see Table I). We see that WEBM with an XY Hamiltonian is the most promising method

	all node interactions		NN interactions	
	XY	XXZ	XY	XXZ
$\frac{T^{(WEBM)}}{\tau_2(11/2)}$	0.041	1.262	0.055	272.228
$\frac{T^{(ELFM)}}{\tau_2(9)}$	0.319	0.144	212.017	21.194

TABLE I: Effect of the inner nodes on the transfer time

of state transfer, namely, such chains allow one to transfer the state over the fixed distance  $L$  (in this case,  $\xi_{1,2} = \xi_{9,10} = 1$ ,  $\xi_{i,i+1} = 1/2$ ,  $2 \leq i \leq 8$ , so that  $L = \sum_{i=1}^9 \xi_{i,i+1} = 11/2$ ) during the shortest time interval  $\frac{T^{(an)}}{\tau_2(11/2)} = 0.041$  see Fig.2a and Table I. However, one has to remember that the XXZ Hamiltonian is the most natural one describing the spin dynamics in the strong external magnetic field. ELFM is better suited for this Hamiltonian. It gives  $\frac{T^{(an)}}{\tau_2(9)} = 0.144$  (in this case,  $\xi_{i,i+1} = 1$ ,  $1 \leq i \leq 9$ , so that  $L = \sum_{i=1}^9 \xi_{i,i+1} = 9$ ), see Fig.5a and Table I.

(ii) Comparing  $T^{(an)}$  with  $T^{(nn)}$  (which are shown in Figs.2a-5a and 2b-5b, respectively) we see that  $T^{(nn)} > T^{(an)}$  in all considered experiments with dipole-dipole interactions, which is reflected in Table II. This observation may be explained as follows. As we have seen, the probability of the end-to-end state transfer  $P(\tau) = |f_{1N}(\tau)|^2$  (where  $f_{1N}$  is defined by eq. (7)) is a superposition of the oscillating functions. Figs.2-5 demonstrate us that the oscillation with the minimal frequency has the maximal amplitude. The transfer time is defined mainly

	WEBM	ELFM
XY	1.334	664.444
XXZ	215.710	146.929

TABLE II: Table of values  $\frac{T^{(nn)}}{T^{(an)}}$

by this oscillation. Note that this frequency is simply related with the minimal eigenvalue  $\lambda_{min}$  of the matrix  $D$  (see eq.(12)) in the case of XY Hamiltonian with WEBM and NN interactions [12]. In fact,  $\lambda_{min} = 0.118$  in this case, so that  $T^{(nn)} \approx \frac{\pi}{\lambda_{min}} = 26.648$ , while the calculated value of  $T^{(nn)}$  is 28.698, see Fig.2b. Remote node interactions increase this minimal frequency, which leads to the decrease in transfer time.

(iii) Let  $T^{(XY)}$  and  $T^{(XXZ)}$  be the transfer times corresponding to the chains governed by the XY and XXZ Hamiltonians respectively. Comparing the transfer times in Fig.2 with the appropriate times in Fig.3 and the transfer times in Fig.4 with the appropriate times in Fig.5 we conclude that

$$\begin{aligned} \text{WEBM : } T^{(XY)} &< T^{(XXZ)}, \\ \text{ELFM : } T^{(XY)} &> T^{(XXZ)}, \end{aligned} \tag{36}$$

see Table III.

	all node interactions	NN interactions
WEBM	0.033	$2.0 \times 10^{-4}$
ELFM	2.212	10.004

TABLE III: Table of values  $\frac{T^{(XY)}}{T^{(XXZ)}}$

(iv) Comparing the transfer times in Figs.2,3 with the appropriate times in Figs.4,5 we conclude that WEBM is preferable for the chains governed by the XY Hamiltonian, while ELFM is suitable for the XXZ Hamiltonian, see Table IV. Since we use different  $L$  for WEBM ( $L = 11/2$ ) and for ELFM ( $L = 9$ ), we have to introduce parameter  $b = (18/11)^3$  in order to get the correct comparison.

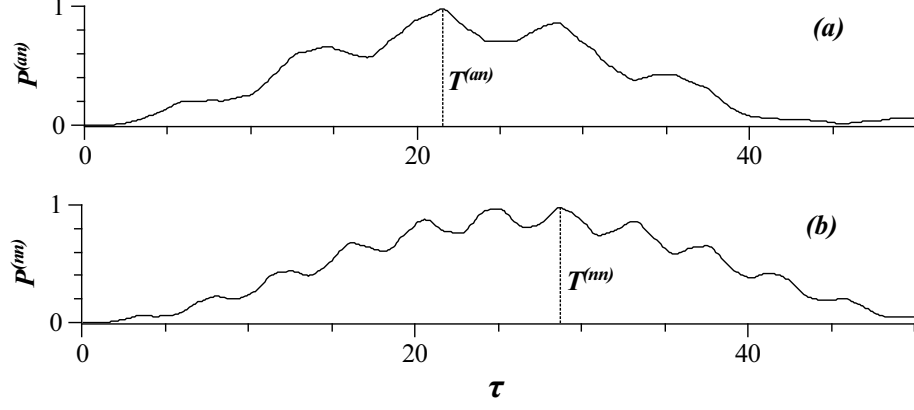


FIG. 2: The end-to-end HPST along the chain of ten nodes governed by the XY Hamiltonian using WEBM;  $\delta = 8$ ,  $\tau_2(11/2) \approx 522.682$ ; (a) all node interactions,  $T^{(an)} = 21.518 < \tau_2$ ,  $P^{(an)} = 0.976$ ; (b) NN interactions,  $T^{(nn)} = 28.698 < \tau_2$ ,  $P^{(nn)} = 0.972$

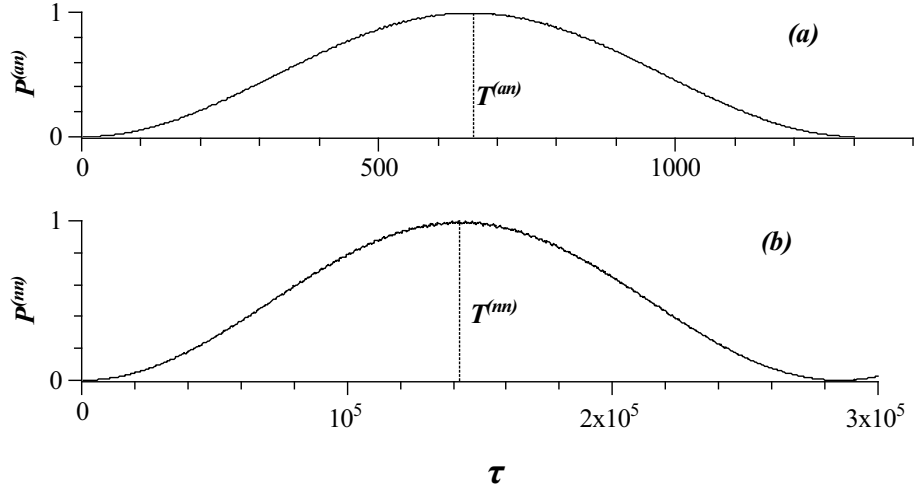


FIG. 3: The end-to-end HPST along the chain of ten nodes governed by the XXZ Hamiltonian using WEBM;  $\delta = 8$ ,  $\tau_2(11/2) \approx 522.682$ ; (a) all node interactions,  $T^{(an)} = 659.630 > \tau_2$ ,  $P^{(an)} = 0.995$ ; (b) NN interactions,  $T^{(nn)} = 142288.896 > \tau_2$ ,  $P^{(nn)} = 1.000$

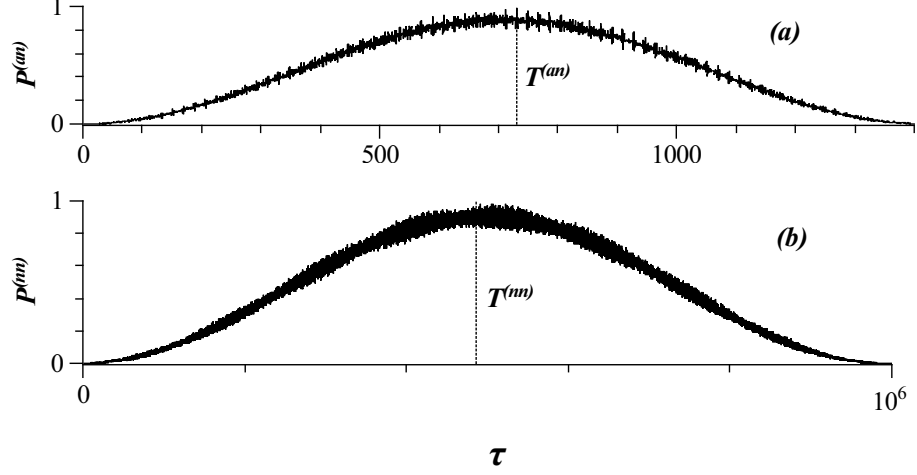


FIG. 4: The end-to-end HPST along the chain of ten nodes governed by the XY Hamiltonian using ELFM;  $\omega = 2.203$ ,  $\tau_2(9) \approx 2290.221$ ; (a) all node interactions,  $T^{(an)} = 730.786 < \tau_2$ ,  $P^{(an)} = 0.985$ ; (b) NN interactions,  $T^{(nn)} = 485566.049 > \tau_2$ ,  $P^{(nn)} = 0.994$

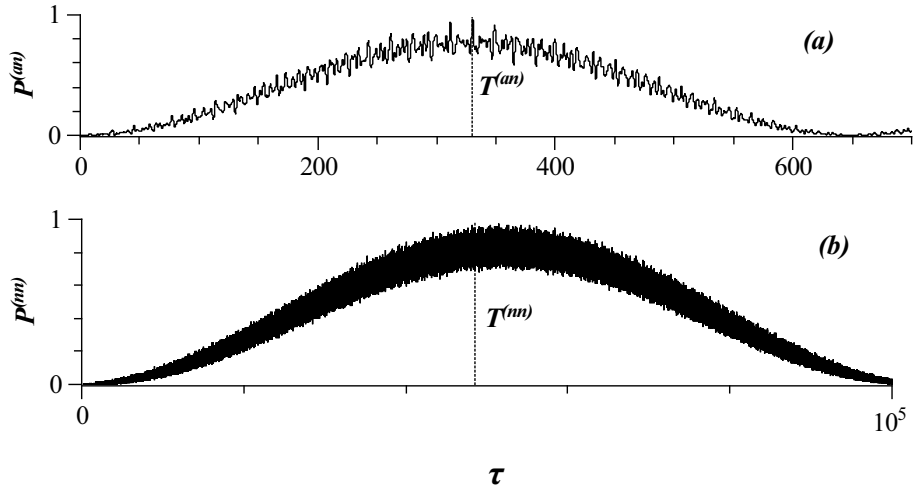


FIG. 5: The end-to-end HPST along the chain of ten nodes with XXZ Hamiltonian and ELFM;  $\omega = 2.651$ ,  $\tau_2(9) \approx 2290.221$ ; (a) all node interactions,  $T^{(an)} = 330.352 < \tau_2$ ,  $P^{(an)} = 0.971$ ; (b) NN interactions,  $T^{(nn)} = 48538.313 > \tau_2$ ,  $P^{(nn)} = 0.973$

	all node interactions	NN interactions
XY	0.129	$2.6 \times 10^{-4}$
XXZ	8.749	12.845

TABLE IV: Table of values  $\frac{T^{(WEBM)}}{T^{(ELFM)}}b$ ,  $b = (18/11)^3$

## V. CONCLUSIONS

We represent an analytical and numerical approaches to the problem of single quantum state transfer along the spin-1/2 chains governed by either XY or XXZ Hamiltonian using WEBM and ELFM. Let us summarize all basic results which have been obtained in this paper.

1. We demonstrate the equivalence of the XY and XXZ Hamiltonians in the problem of the single excited state transfer along the spin chain with nonzero Larmor frequencies, i.e. one can transform the XY Hamiltonian into XXZ Hamiltonian and vice-versa taking proper Larmor frequencies.
2. We have derived the analytical expressions for the end-to-end state transfer probabilities along the chain governed by the XY Hamiltonian using a combination of WEBM and ELFM with NN interactions, which is most applicable to the interactions which are quickly decreasing with an increase in the distance (like the exchange interactions).
3. Numerical simulations of the spin dynamics along the ten-node chain with dipole-dipole interactions allow us to compare the end-to-end transfer times corresponding to the different Hamiltonians, different methods of state transfer, and different types of interactions, see Tables I-IV. In particular, it is shown that inner nodes are not always usefull in the process of state transfer over the fixed distance  $L$ .

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